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Squeezing a wave packet with an angular-dependent mass

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Abstract
We present a new effect of position-dependent mass (PDM) systems: the possibility of creating squeezed wave packets at the partial revival times. We solve exactly the PDM Schrödinger equation for the two-dimensional quantum rotor with two effective masses $\mu(\theta)$, both free and interacting with a uniform electric field, and present their energy eigenvalues and eigenfunctions in terms of Mathieu functions. For the first one, in order to squeeze the wave packet it is necessary to apply an electric field; for the second one such an effect can be achieved without the field.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The two-dimensional quantum rotor in quantum mechanics, despite being a simple and well-known system, can be used to model newly proposed devices such as molecular motors [1]. The first experimental realization of quantum rotors was made by Yasutomi and collaborators [2] using molecular structures over a gold substrate. Another kind of quantum rotor, the so-called hindered rotors were introduced by Gadzuk [3], i.e., rotors confined to move within a conical region $0 \leqslant \theta \leqslant \theta_{\text{max}}$ instead of the whole $[0, 2\pi]$. These theoretical systems were used to model adsorbed dipolar molecules [4], which in turn are good candidates to create entangled states between coupled rotors [5]. Such dipoles are the key feature of a possible design of quantum computers [6]. Other applications were proposed by Rogovin and Nagel [7] who showed an interesting analogy between the quantum rigid rotor interacting with a dc electric field and an ultrasmall Josephson junction.

On the other hand, in the last few years researchers have been pursuing a quest for exact solutions of the Schrödinger equation where the effective mass is allowed to depend on the position, i.e., $\mu = \mu(x, y, z)$, as well as on time $\mu = \mu(t)$, or even both position and time [8]. Moreover, there exists a long-standing ordering problem, namely, Hamiltonians for position-dependent mass (PDM) systems can be written in terms of three parameters α, β, γ, as the so-called von Roos [9] Hamiltonian that reads

$$H = \frac{1}{2}\left(M^{\alpha} p_i M^{\beta} p_i M^{\gamma} + M^{\gamma} p_i M^{\beta} p_i M^{\alpha}\right) + U, \tag{1}$$

with U denoting the potential, p the momentum operator, $\mu(\vec{r})$ the dimensionless position-dependent mass, $M(\vec{r}) = m_0\mu(\vec{r})$, and the parameters being such that $\alpha + \beta + \gamma = -1$. In a previous paper [10] we studied the 2D infinite circular well and presented some arguments that point out that Zhu and Kroemer [11] (ZK) parameters, $\alpha = \gamma = -1/2, \beta = 0$, are those physically allowed. Recently Souza Dutra and Oliveira studied 2D PDM systems interacting with a magnetic field [12] and Mustafa and Mazharimousavi [13] presented five new sets that are also good physical candidates when one has a singular mass distribution. Those are the sets of parameters that result in self-adjoint Hamiltonians [14, 15].

The first motivation to study PDM systems is purely theoretical: to investigate which Hamiltonian is realized or can be realized in Nature and to find new exact solutions for the Schrödinger equation [16–19]. The second motivation is related to the several applications such systems can have: modeling of scattering on abrupt heterostructures [20], quantum dots where the mass depends on the radial distance [10, 21], supersymmetric quantum mechanics [22], the possibility of creating the astonishing cloaking effect [23] controlling the effective mass, as well as applications in other condensed matter problems [24–26].

However, as far as we know, there is not a study on the Schrödinger equation for the angular-dependent mass, $\mu = \mu(\theta)$. We propose to fill this gap by solving the Schrödinger equation for a quantum rotor with angular-dependent mass $\mu(\theta)$. We choose an angular mass distribution which is real and periodic in the region $0 \leqslant \theta < 2\pi$, namely $\mu(\theta) = \mu_1 - \mu_2^2\cos(b\theta)$ where $b = 1$ or $b = 2$. Such mass distribution was chosen because it covers several cases, depending on the values μ_1 and μ_2 can take: the constant mass quantum rotor $\mu_2 = 0$, a quantum rotor with a slightly varying mass with $\mu_2^2 \ll \mu_1$, as well as a rotor where μ_1 and μ_2^2 are of similar size, so that the mass has large variations.

The outline of our paper is the following: in section 2 we propose the problem of angular-dependent mass and present the corresponding Schrödinger equation for a single particle interacting with a uniform electric field and its exact eigenfunctions—written in terms of Mathieu functions—and the set transcendental equations that provide the energy eigenvalues for two mass distributions, given by (4) and (15). In section 3 we analyze the long-term time evolution using a Gaussian wave packet and the effects of revival [27]. Partial revivals are suppressed and using this fact we show how to create a squeezed wave packet, applying an electric field in the first case and with no electric field in the second. Section 4 is devoted to our conclusions.

2. The Schrödinger equation for a rigid rotor with $\mu = \mu(\theta)$

Let us study the case where one has a quantum particle free to rotate around the origin but always the distance R, which we take to be $R = 1$, from it and its mass depends on the angle θ. Following Quesne [14] we write the Hamiltonian for a single particle when its mass depends on the angular position:

$$H^{(1)}_{pdm} = -\frac{1}{\mu}\frac{\partial^2}{\partial\theta^2} + \frac{\mu_\theta}{\mu^2}\frac{\partial}{\partial\theta} + \frac{(1+\beta)\mu_{\theta\theta}}{2\mu^2} - \frac{f(\alpha,\beta)\mu_\theta^2}{\mu^3}, \tag{2}$$

and the corresponding time-independent Schrödinger equation

$$\phi_n''(\theta) - \frac{\mu_\theta \phi_n'(\theta)}{\mu} - \frac{(1+\beta)\mu_{\theta\theta}\phi_n(\theta)}{2\mu} + f(\alpha,\beta)\frac{\mu_\theta^2\phi_n(\theta)}{\mu^2} - \mu[E_n - U(\theta)]\phi_n(\theta) = 0, \quad (3)$$

where we use atomic Rydberg units $\hbar = m_0 = 1$, $U(\theta)$ is the potential, E_n are the energy eigenvalues, $\mu = \mu(\theta)$ is the dimensionless particle effective mass, μ_θ and $\mu_{\theta\theta}$ are its first and second derivatives respectively and

$$f(\alpha,\beta) = \alpha(\alpha+\beta+1) + \beta + 1,$$

the parameters α and β represent the way one order the kinetic operator [13, 14]. The same Hamiltonian can be interpreted as the usual kinetic term $\vec{\nabla}(\mu^{-1})\vec{\nabla}$ plus a potential which depends on the mass and its derivatives. The problem in question has a non-central potential, which as far as we know was not studied before [28, 29]. For physical reasons the function $\mu(\theta)$ must be periodic and real. So let us try two functions that satisfy these properties.

2.1. Mass distribution $\mu(\theta) = \mu_1 - \mu_2^2\cos\theta$

Let the effective mass be

$$\mu(\theta) = \mu_1 - \mu_2^2\cos\theta, \quad (4)$$

where $\mu_1 > \mu_2^2 > 0$. Substituting it into equation (3), using [11] the parameters of Zhu and Kroemer $\alpha = -1/2$ and $\beta = 0$, as well as taking $U = V/\mu$, being

$$V(\theta) = -\mu_e\varepsilon\cos\theta, \quad (5)$$

where without loss of generality we take the rotor electric charge to be $q = 1$, μ_e is the dipole momentum magnitude, the electric field is taken to be uniform with magnitude ε and oriented along the Ox axis, one arrives at

$$\phi_n''(\theta) + V_\mu(\mu_2,\theta) + \mu_e\varepsilon\phi_n(\theta)\cos\theta = \mu E_n\phi_n(\theta), \quad (6)$$

where the second term bears the angular dependence of mass

$$V_\mu(\mu_2,\theta) = -\frac{\mu_2^2}{\mu}\left[\phi_n'(\theta)\sin\theta + \frac{1}{4\mu}\left(2\mu_1\cos\theta + \mu_2^2\cos^2\theta - 3\mu_2^2\right)\phi_n(\theta)\right], \quad (7)$$

that is, if we let $\mu_2 = 0$ we obtain the usual Schrödinger equation of a 2D rigid rotor interacting with an uniform electric field [30]. If one tries the usual ansatz,

$$\phi_n(\theta) = \sqrt{\mu(\theta)}F_n(\theta), \quad (8)$$

then the Schrödinger equation becomes the well-known Mathieu equation

$$F_n''(\theta) + \left[-\mu_1E_n + \left(\gamma + \mu_2^2E_n\right)\cos\theta\right]F_n(\theta) = 0, \quad (9)$$

where $\gamma = \mu_e\epsilon$, and the eigenfunctions can be written in terms of even Mathieu functions [31]

$$\phi_{2r}^{\text{even}}(\theta) = c_1\sqrt{\mu_1 - \mu_2^2\cos\theta}\,\text{ce}_{2r}\left(-4\mu_1E_{2r}, -2\mu_2^2E_{2r} - 2\gamma; \frac{\theta}{2}\right), \quad (10)$$

and odd Mathieu functions

$$\phi_{2r+2}^{\text{odd}}(\theta) = c_2\sqrt{\mu_1 - \mu_2^2\cos\theta}\,\text{se}_{2r+2}\left(-4\mu_1E_{2r+2}, -2\mu_2^2E_{2r+2} - 2\gamma; \frac{\theta}{2}\right), \quad (11)$$

with $r = 0, 1, 2, \ldots$ and c_1 and c_2 being normalization constants. Now we have to impose periodicity for the eigenfunction $\phi_n(\theta)$ over $\theta \to \theta + 2\pi$. Mathieu's functions $\text{ce}_n(a, q; z)$ and $\text{se}_n(a, q; z)$ are periodic only for a careful choice of the parameters a and q, see [31, 32], namely when a is a characteristic value associated with q, i.e., $a_n(q)$ for $\text{ce}_n(\cdot)$ function and

$b_n(q)$ for $\mathrm{se}_n(\cdot)$ function. Given a certain value of q there are infinite characteristic values $a_n(q)$ for which Mathieu function $\mathrm{ce}_n(\cdot)$ is periodic. However, it can have period equal to π or 2π, then one must verify which values b can take in order to satisfy this requirement. The same holds for the Mathieu function $\mathrm{se}_n(\cdot)$.

However, in order to calculate the energy eigenvalues E_n we face a problem. While in the simple pendulum, with mass M and length L, one also has eigenfunctions written as Mathieu functions and $q = -2MgL/(\hbar^2/2I)$, being $I = ML^2$, one can generate the sets a_n and b_n straightforwardly (numerically) since q does not depend of the energy. The key point is to recall that since $q < 0$ one must perform a transformation in order to maintain Mathieu's function periodicity [32], eventually the energy eigenvalues can be written as $E_n^{\mathrm{even}} \propto a_n(q)$ as well as $E_n^{\mathrm{odd}} \propto b_n(q)$. It is not so in our example: here the parameter which plays the role of q depends on the energy eigenvalues, namely $q = -2(\mu_2 E_n + \gamma)$, as well as the parameters which play the role of $a_n = -4\mu_1 E_n$ and $b_n = -4\mu_1 E_n$ do. So we must deal with two transcendental equations involving Mathieu characteristic values, namely, for the eigenvalues a_{2r} of the even function

$$a_{2r}\left(-2\mu_2^2 E_{2r} - 2\gamma\right) + 4\mu_1 E_{2r} = 0, \tag{12}$$

and for the eigenvalues b_{2r+2} of the odd function

$$b_{2r+2}\left(-2\mu_2^2 E_{2r+2} - 2\gamma\right) + 4\mu_1 E_{2r+2} = 0, \tag{13}$$

where $r = 0, 1, 2, \ldots$.

Taking for example $\mu_1 = 2$ and $\mu_2 = 1$ we find only *negative* values for the energy, that is, differently from the simple pendulum the parameter q of the PDM quantum rotor is positive and we do not need to carry out a transformation in order to ensure periodicity. The quantum rotor with an angular-dependent mass given by equation (4) has only bound states. The potential despite having a finite depth can retain an infinite number of bound states, such property was noticed by Souza Dutra and Oliveira in [12] in another system. Solving these transcendental equations can seem a hard numerical task, since the characteristic values of Mathieu functions can be written as continued fractions or a Hill's determinant. We use the first method as well as Mathematica 6.0.3, which has two built-in functions `MathieuCharacteristicA` and `MathieuCharacteristicB` and can turn this problem into an easy one. Using the first approach we have to deal with, after truncating and simplifying a huge continued fraction, a polynomial equation of order 100, at least. It is important to carry out this numerical calculation with high accuracy, because the Mathieu characteristic values are well known to be hard to obtain precisely. This problem is highly simplified if one uses the two built-in functions of Mathematica.

The energy eigenvalues which render the eigenfunctions periodic are the solutions of equations (12) and (13). We label them $E_0^{\mathrm{even}}, E_2^{\mathrm{even}}, E_4^{\mathrm{even}}, \ldots$ for the even wavefunctions, and $E_2^{\mathrm{odd}}, E_4^{\mathrm{odd}}, E_6^{\mathrm{odd}}, \ldots$ for the odd eigenfunctions. In table 1 we present the first 12 energy eigenvalues for the specific rotors studied in the next section.

Since the Mathieu functions are orthogonal, that is,

$$\int_0^{2\pi} \mathrm{ce}_n(a, q; \theta/2)\mathrm{ce}_m(a, q; \theta/2)\,\mathrm{d}\theta = \pi\delta_{nm}, \tag{14}$$

where δ_{nm} is the Kronecker delta, we have a complete set of orthonormal functions, as long as we adjust c_1 and c_2 accordingly. The odd functions $\mathrm{se}_n(\cdot)$ also satisfy a similar condition, and a product between $\mathrm{ce}_n(\cdot)$ and $\mathrm{se}_m(\cdot)$ vanish as well. If one remembers that even (odd) Mathieu functions are defined as a series of cosine (sine) functions [31, 32] — e.g.,

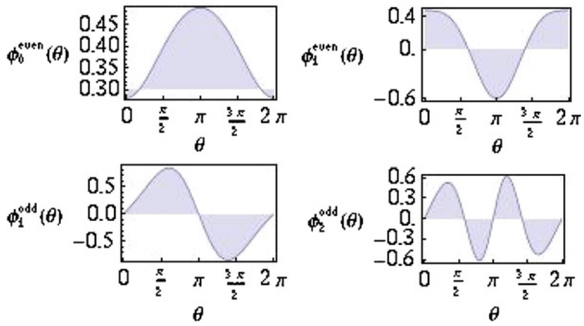

Figure 1. Plot of the first two even and odd eigenfunctions of period π of the Schrödinger equation with $\mu(\theta)$ given by equation (4) where $\mu_1 = 2, \mu_2 = 1$.

Table 1. Energy eigenvalues for mass distribution (4) where the parameters are $\mu_1 = 2, \mu_2 = \sqrt{3/2}, \gamma = 2$.

n	E_n^{even}	E_n^{odd}
1	−0.636 810 284 666 593 922 0502	−0.478 704 839 868 129 477 2005
2	−2.289 147 381 289 625 378 277	−2.113 614 654 337 335 873 925
3	−4.994 722 380 954 351 045 232	−4.854 397 989 206 943 965 278
4	−8.782 587 789 712 752 986 608	−8.684 366 377 191 600 386 500
5	−13.662 681 934 516 946 898 65	−13.598 600 152 376 665 842 13
6	−19.636 588 261 387 332 958 29	−19.596 548 400 589 765 669 19
7	−26.703 386 675 477 101 741 77	−26.679 091 141 082 404 217 06
8	−34.861 776 508 661 415 715 80	−34.847 344 039 871 604 083 06
9	−44.110 681 715 942 183 464 70	−44.102 245 383 202 522 138 59
10	−54.449 342 252 800 606 529 33	−54.444 472 981 934 816 347 74
11	−65.877 260 160 342 541 230 02	−65.874 478 315 209 546 348 86
12	−78.394 123 171 363 564 168 59	−78.392 547 225 400 646 476 96

$\mathrm{ce}_{2r}(a, q, \theta) = \sum_{k=0}^{\infty} A_{2k}(q)\cos(2k\theta)$ —, then it is easy to observe, see equations (8), (10) and (11), that the terms which have the additional factor $\cos(\theta)$,

$$\int_0^{2\pi} \mathrm{ce}_n(a, q; \theta/2)\mathrm{ce}_m(a, q; \theta/2)\cos(\theta)\,\mathrm{d}\theta = 0,$$

also leads to zero.

We plot the first two normalized eigenfunctions, even and odd, in figure 1.

2.2. *Mass distribution* $\mu(\theta) = \mu_1 + \mu_2^2\cos 2\theta$

Let us turn to another mass distribution, namely, the one where

$$\mu(\theta) = \mu_1 + \mu_2^2\cos 2\theta, \tag{15}$$

from equation (2) we write the PDM Schrödinger equation with a different external potential, namely $V(\theta) = \mu_e\varepsilon\cos^2\theta$ which can represent a nonresonant laser field [33] with constant envelope,

$$\phi_n''(\theta) + W_\mu(\mu, \theta) + \mu_e\varepsilon\phi_n(\theta)\cos^2\theta = \mu E_n\phi_n(\theta), \tag{16}$$

where the function $W_\mu(\mu,\theta)$ is given by

$$W_\mu(\mu,\theta) = \frac{\mu_2^2}{\mu}\left\{(2\sin 2\theta)\phi_n'(\theta) - \frac{1}{2\mu}\left[-4\mu_1\cos 2\theta + \mu_2^2(-5+\cos 4\theta\right]\phi_n(\theta)\right\}, \tag{17}$$

using the same ansatz (8) of section 2.1 we obtain a Mathieu equation

$$F_n''(\theta) + \left[-\mu_1 E_n + \frac{\gamma}{2} + \left(\frac{\gamma}{2} - \mu_2^2 E_n\right)\cos(2\theta)\right]F_n(\theta) = 0, \tag{18}$$

usually written as $y''(x)+(a-2q\cos 2x)y(x) = 0$ with different parameters, $a = -\mu_1 E_n + \frac{\gamma}{2}$ and $q = \frac{\gamma}{2} - \mu_2^2 E_n$. Observe that both parameters depend on the energy eigenvalues E_n, so in order to obtain periodic Mathieu functions we have to solve a set of transcendental equations such as (12) and (13). The Schrödinger equation is satisfied by both Mathieu functions of period equal to π,

$$\chi_{2r}^{\text{even}}(\theta) = c_1\sqrt{\mu_1 + \mu_2^2\cos 2\theta}\,\text{ce}_{2r}\left(-\mu_1 E_{2r} + \frac{\gamma}{2}, \frac{\mu_2^2 E_{2r}}{2} - \frac{\gamma}{4}; \frac{\theta}{2}\right), \tag{19}$$

and

$$\chi_{2r+2}^{\text{odd}}(\theta) = c_2\sqrt{\mu_1 + \mu_2^2\cos 2\theta}\,\text{se}_{2r+2}\left(-\mu_1 E_{2r+2} + \frac{\gamma}{2}, \frac{\mu_2^2 E_{2r+2}}{2} - \frac{\gamma}{4}; \frac{\theta}{2}\right), \tag{20}$$

as well as for Mathieu functions of period 2π,

$$\chi_{2r+1}^{\text{even}}(\theta) = d_1\sqrt{\mu_1 + \mu_2^2\cos 2\theta}\,\text{ce}_{2r+1}\left(-\mu_1 E_{2r+1} + \frac{\gamma}{2}, \frac{\mu_2^2 E_{2r+1}}{2} - \frac{\gamma}{4}; \theta\right), \tag{21}$$

and

$$\chi_{2r+1}^{\text{odd}}(\theta) = d_2\sqrt{\mu_1 + \mu_2^2\cos 2\theta}\,\text{se}_{2r+1}\left(-\mu_1 E_{2r+1} + \frac{\gamma}{2}, \frac{\mu_2^2 E_{2r+1}}{2} - \frac{\gamma}{4}; \theta\right), \tag{22}$$

with $r = 0, 1, 2, \ldots$ and the normalization constants c_1, c_2, d_1 and d_2. In other words both sets satisfy the constraint $\chi(\theta) = \chi(\theta + 2\pi)$.

The characteristic values which give the correct periodic behavior are a_{2r+1} and b_{2r+1} and can be obtained by solving the equations

$$a_{2r+1}\left(\frac{\mu_2^2 E_{2r+1}}{2} - \frac{\gamma}{4}\right) + \mu_1 E_{2r+1} - \frac{\gamma}{2} = 0, \tag{23}$$

and

$$b_{2r+1}\left(\frac{\mu_2^2 E_{2r+1}}{2} - \frac{\gamma}{4}\right) + \mu_1 E_{2r+1} - \frac{\gamma}{2} = 0, \tag{24}$$

the first equation provides the energy eigenvalues E_{2r+1}^{even} and the second one E_{2r+1}^{odd}. Similar equations hold for a_{2r} and b_{2r+2} and provide other two sets of energy eigenvalues E_{2r}^{even} and E_{2r+2}^{odd}. Both sets of Mathieu functions, those of period π as well as 2π, form an orthonormal complete set and we can write any well-behaved function in this basis. The first two eigenfunctions of period 2π are plotted in figure 2.

When $0 < b < 1$ the mass distribution is not single valued over $\theta \in [0, 2\pi]$ and the Mathieu functions can be periodic only over π or 2π so these effective masses have no physical interest. The same occurs for non-integer $b > 1$: the Mathieu functions can have only period π or 2π, so the only values left, besides those we already studied, are $b = 3, 4, 5, \ldots$: odd ones will lead to the first case, i.e., wavefunctions written as $\phi_n^{\text{even,odd}}(\theta)$; the even ones can be written as $\phi_n^{\text{even,odd}}(\theta)$ and $\chi_n^{\text{even,odd}}(\theta)$.

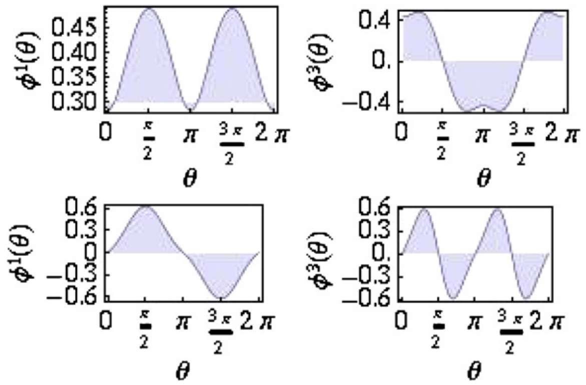

Figure 2. Plot of the first two even (upper boxes) and odd (lower boxes) eigenfunctions of period 2π of the Schrödinger equation with $\mu(\theta)$ given by equation (15) and $\mu_1 = 2, \mu_2 = 1$.

3. Application

As an application of the previous results, let us investigate the behavior of the PDM quantum rotor studying its time evolution. In previous papers [10, 34] we analyzed the free propagation of a Gaussian wave packet (GWP) in 1D and 2D (circular) infinite potential wells, as well as interacting with a parabolic potential in the two-dimensional case, both when the mass depends on the radial position. While the well-known effect of wave packet revival [27] takes place, on the other hand, partial revival occurs not as simple smaller copies, with the whole probability divided among the number of these small packets, but as a single and squeezed copy of the original GWP. In this section we will present both effects for the quantum rotor with an angular-dependent mass.

The normalized GWP can be written as

$$\Psi_0(\theta) = \frac{\exp\left(\Delta_0^{-2}\right)}{\sqrt{2\pi I_0\left(2\Delta_0^{-2}\right)}} \exp\left[-\mathrm{i}\vec{p}_0 \cdot \vec{r} - \frac{1}{2\Delta_0^2}(\vec{r} - \vec{r}_0)^2\right], \tag{25}$$

where Δ_0 is related to the Gaussian width, $I_\nu(z)$ is the modified Bessel function, $\vec{p}_0$ and $\vec{r}_0$ are the initial momentum and position, respectively, and the vector position is $\vec{r}$. We must write the complete time-dependent wavefunction $\psi(\theta, t)$ as a superposition of eigenfunctions, namely,

$$\psi(\theta, t) = \sum_{n=0}^{\infty} c_n F_n(\theta) \exp(-\mathrm{i}E_n t), \tag{26}$$

where $\psi(\theta, 0) = \Psi_0(\theta, 0)$ in accordance with (25), $F_n(\theta)$ represents either $\left\{\phi_n^{\text{even,odd}}(\theta)\right\}$ or $\left\{\chi_n^{\text{even,odd}}(\theta)\right\}$ above and E_n are the associated energy eigenvalues. Taking the rotor radius to be unity, the initial position to be $\vec{r}_0 = \hat{j}$ where $\hat{j}$ is the usual unit vector in the Oy direction, which results in an initial angle $\theta_0 = \pi/2$, as well as its initial momentum $\vec{p}_0$ to be zero, then we have a wave packet of the form

$$\Psi_0(\theta, 0) = \mathcal{N} \exp(-\mathcal{A} \sin\theta), \tag{27}$$

where $\mathcal{N}$ is a normalization constant, $\mathcal{A}$ is a factor which depends on the initial position, momentum etc, and the Fourier coefficients, c_n, can be calculated in the usual way, since the eigenfunctions form a complete orthonormal set

$$c_n = \int_0^{2\pi} \Psi_0(\theta, 0) F_n^*(\theta)\, \mathrm{d}\theta, \tag{28}$$

where $F_n^*(\theta)$ represents the complex conjugate of the eigenfunction in question. Unfortunately, as far as we know, this kind of integral is not tabulated and we must resort to numerical integration of (28). In order to construct a GWP with probability almost equal to 1, it is sufficient to have typically the first 22 eigenstates for both $\phi_{2r}^{\mathrm{even}}(\theta)$ and $\phi_{2r+2}^{\mathrm{odd}}(\theta)$; in the second case the first nine eigenfunctions for each of the four sets $\phi_{2r}^{\mathrm{even}}(\theta)$, $\phi_{2r+2}^{\mathrm{odd}}(\theta)$, $\chi_{2r+1}^{\mathrm{even}}(\theta)$ and $\chi_{2r+1}^{\mathrm{odd}}(\theta)$ are enough to have a precision better than 10^{-8} for the GWP (25), i.e., $\sum_n |c_n|^2 = 1 - \mathcal{O}(10^{-8})$. It is worth mentioning that such numerical integration must also be carried out with high precision, since Mathieu functions are oscillating ones (in Mathematica we set `WorkingPrecision` to 22, as well as `MaxRecursion` to at least 35 and `MaxErrorIncreases` to 15 000).

With the Fourier coefficients calculated we can write the time-dependent wavefunction $\psi(\theta, t)$ and investigate how $|\psi(\theta, t)|^2$ evolves with time. The usual rigid rotor has a well-known dynamics: its Fourier coefficients can exactly be calculated in terms of Bessel functions; at time $T_{\mathrm{rev}} = 4N\pi$, being $N = 1, 2, 3, \ldots$ the GWP returns to its initial position, the well-known revival effect [27]. At one half (one quarter) of this revival time, $t = 2\pi$ $(t = \pi)$, two (four) smaller identical copies of the GWP occur, and so forth. Such effect is called partial revival. This is not so when the mass is allowed to depend on the position as we will see in the next section.

3.1. Squeezing a wave-packet

In order to squeeze [35, 36] a wave packet, e.g., transforming the original probability density, usually written as a Gaussian wave packet, into another Gaussian with greater amplitude but smaller width, one in general applies an external time-dependent potential. Delgado and Mielnik worked on squeezing effect using time-dependent magnetic fields [37]. Grigorenko [38] applied optimal control techniques in order to achieve the squeezing effect of the particle inside an infinite well using an external potential written as a series oscillating terms. We follow instead a suggestion of Brown and Rabitz [39]: the properties of the materials could produce desired predetermined effects or control the time evolution of physical systems. This is for instance the key idea of the so-called cloaking effect of matter waves [23]. Here, we can carefully choose three parameters: μ_1 and μ_2 which define our mass distribution, and the field strength γ. The proper interference pattern, the squeezing effect, will follow without changing the external field configuration (or even without this field) and will repeat itself periodically. This effect can be seen again according to the quantum carpet shown in figure 3.

We can observe the probability density rebuilding itself in white regions which represent high probability densities, low probabilities are represented by dark blue ones. The squeezing effect occurs because the GWP propagates faster in the region of smaller mass, on the other hand, it slowly penetrates the region where the mass is higher. At these times, namely $T_{\mathrm{rev}}/2$ and its multiples, the wave packet already managed to enter this region and the entire probability density rebuilds itself. So when the effective mass distribution is not uniform a Gaussian wave packet can be *naturally* squeezed, i.e., the background where it propagates greatly contributes to create a very narrow wave packet.

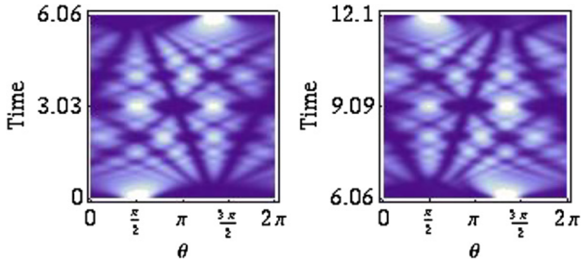

Figure 3. Density plot of $|\psi(\theta, t)|^2$, the well-known quantum carpet: in the left box we plot from $t = 0$ to $t = T_{\text{rev}}/2$; in the right box from $t = T_{\text{rev}}/2$ to $t = T_{\text{rev}}$, where $T_{\text{rev}} = 12.126$, $\mu_1 = 2$, $\mu_2 = 1$ and $\gamma = 0$. Observe that instead of two concentrated probability densities, equally spaced at $T_{\text{rev}}/2$ this system presents only one, a squeezed wave packet revival.

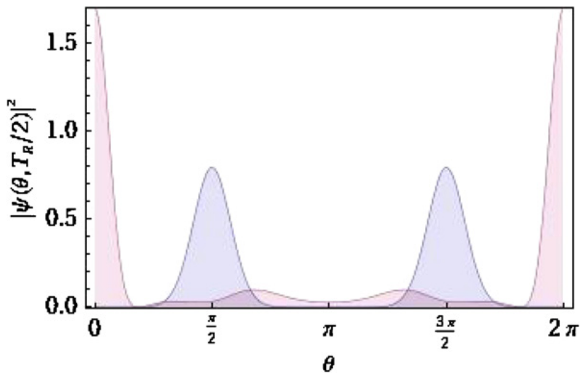

Figure 4. The squeezed wave packet takes place at $T_{\text{rev}}/2 = 0.7231$, and the parameters are $\Delta_0 = 1/2\sqrt{2}$, $\gamma = 2$, $\theta_0 = \pi/2$, and the initial momentum equal to zero. This field strength must be carefully chosen since a slight change destroys the squeezing effect. The blue curve represents the initial GWP and the pink curve shows four squeezed wave packets. Such an effect will repeat itself at multiples of the above time.

Consider the initial wave packet to be split into two equal parts namely, a GWP centered at $\theta = \pi/2$ and another one centered at $\theta = 3\pi/2$ both with equal probability,

$$\Phi_1(\theta) = \frac{1}{\sqrt{2}} \exp\left(-\frac{1 - \sin\theta}{\Delta_0^2}\right) + \frac{1}{\sqrt{2}} \exp\left(-\frac{1 + \sin\theta}{\Delta_0^2}\right), \tag{29}$$

see equation (25). Let the mass distribution be

$$\mu(\theta) = 2 + \tfrac{3}{2}\cos\theta, \tag{30}$$

their initial momentum is zero, $\Delta_0 = 1/2\sqrt{2}$ and $\gamma = 2$. From the previous section 2A we know that the eigenfunctions are given by (10) and (11). In order to squeeze the wave packet the electric field is necessary, here we take $\gamma = 2$. A slight variation of γ destroys the squeezing effect we want to generate.

In figure 4 we can observe that both parts of the initial packet have an initial amplitude equal to 0.8 (blue curve). After an interval of time $T_R = 291.26$ the probability density returns to its initial form, however, at half of this time both packets manage to enter the region of higher mass—the propagation in the region of smaller mass is faster than in the higher mass region—and interfere with the other one. The result is a squeezed wave packet, as we can see

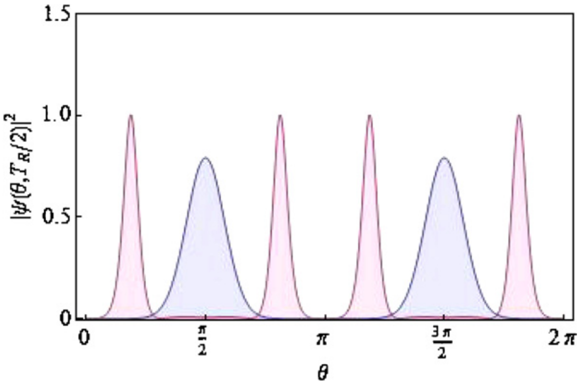

Figure 5. Now $T_{\rm rev}/2 = 145.6343$, the parameters are the same of the previous figure, except the field strength that is equal to zero. Blue curve represent the initial GWP and pink curve show four squeezed wave packets.

in the figure, centered at $\theta = 0$, this single packet has an amplitude equal to approximately 1.7 (pink curve), twice the initial one. Approximately 78% of it is concentrated around $\theta = (0.0 \pm 0.6)$ rad.

A similar effect also takes place for the second mass distribution (15). If we consider the initial wave packet to be split into two equal Gaussians

$$\Phi_2(\theta) = \frac{1}{\sqrt{2}} \exp\left(-\frac{1 - \sin\theta}{\Delta_0^2}\right) - \frac{1}{\sqrt{2}} \exp\left(-\frac{1 + \sin\theta}{\Delta_0^2}\right), \tag{31}$$

with $\mu(\theta) = 2 + \frac{3}{2}\cos(2\theta)$ and the second one with an extra negative phase. Surprisingly the electric field here is not important, even with zero field strength $\gamma = 0$ we can observe four narrow wave packets at the time in which would occur the partial revival, $T_R/2 = 0.7231$, see figure 5.

Systems where the mass depends on the position produce naturally very narrow probability densities, sharper than a Gaussian, at these special times. This is an example where the properties of the material—the form of $\mu(\theta)$ as well as the specific values of μ_1 and μ_2—can drive a wave packet without an external potential [39].

4. Conclusion

We presented the exact solution for the Schrödinger equation of a quantum rigid rotor which has an effective mass depending on the angular position and interacts with a uniform electric field, specifically when the distributions are given by $\mu(\theta) = \mu_1 - \mu_2^2\cos(\theta)$, or $\mu(\theta) = \mu_1 + \mu_2^2\cos(2\theta)$. These exact eigenfunctions are written in terms of Mathieu functions of period π or 2π depending on the value of b. We used the obtained results to investigate the long-term time evolution of a Gaussian wave packet studying its partial and full revival times. The suppression of the partial revival effect leads to a squeezing of the wave packet at $T_{\rm rev}/2$ for both mass distributions (4) and (15)—applying a uniform electric field in the first case, and in the second the field is not necessary—so we conclude that position-dependent mass systems can produce naturally squeezed wave packets.

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